Conic Programming in GAMS

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Direction

- What this talk *is* about
  - Overview: the class of conic programs
  - Conic constraints and examples using GAMS
  - Solvers in GAMS and numerical results
  - CONE World - a forum for conic programming

- What this talk *is not* about:
  - Conic programming algorithms
  - Detailed applications

Overview

What are conic programs?

- Generalized linear programs with the addition of nonlinear convex cones
- Class includes, for example,
  - Linear program (LP)
  - (Convex) Quadratic program (QP)
  - (Convex) Quadratically constrained QP (QCQP)
- Recently much activity in this area!

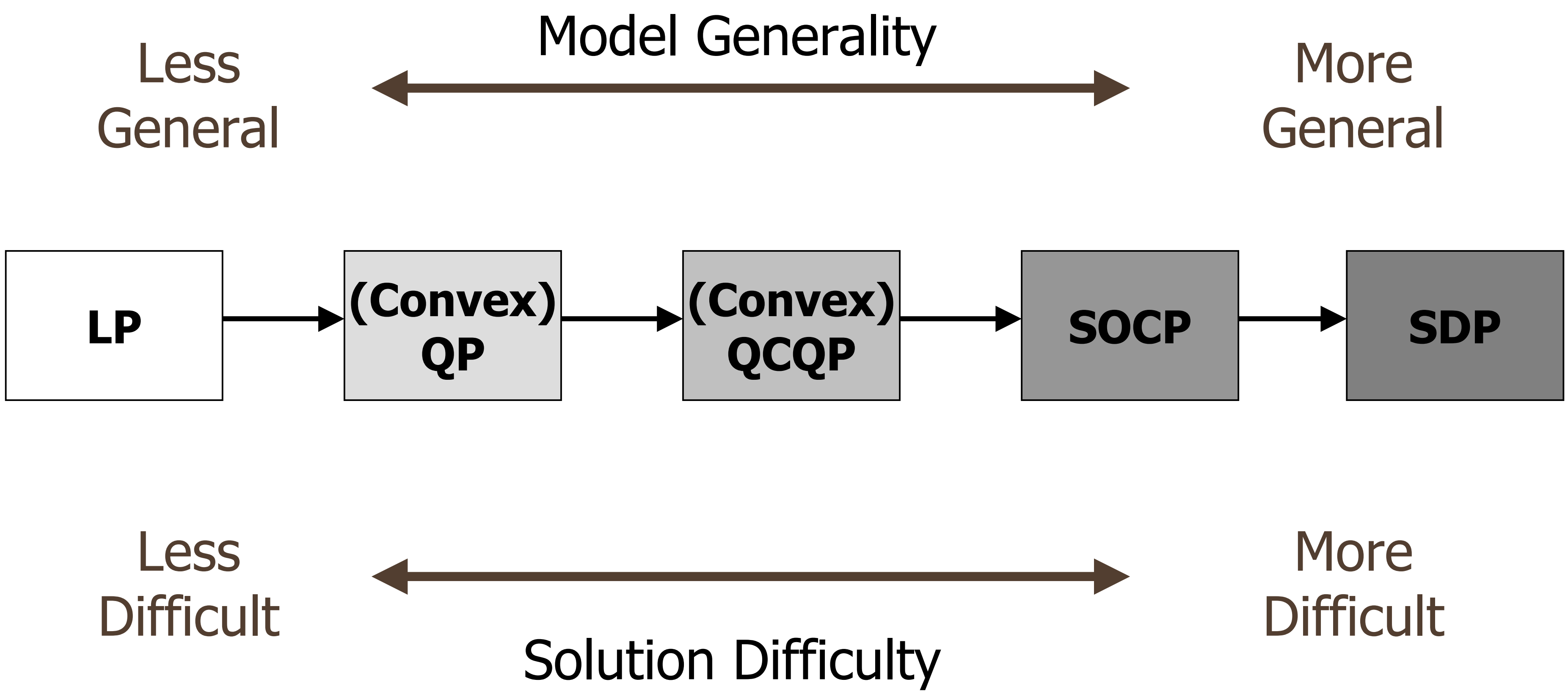
Areas and Applications

Conic programming used for:

- Engineering
  - Truss topology design
  - FIR filter design
- Finance
  - Portfolio optimization
- Statistics and Numerical Linear Algebra
  - Robust linear programming
  - Norm minimization problems

7th DIMACS Implementation Challenge on SDP and Second Order Cone Programming (SOCP)

Modeling and Solving



Cone Programs

- General form of conic program

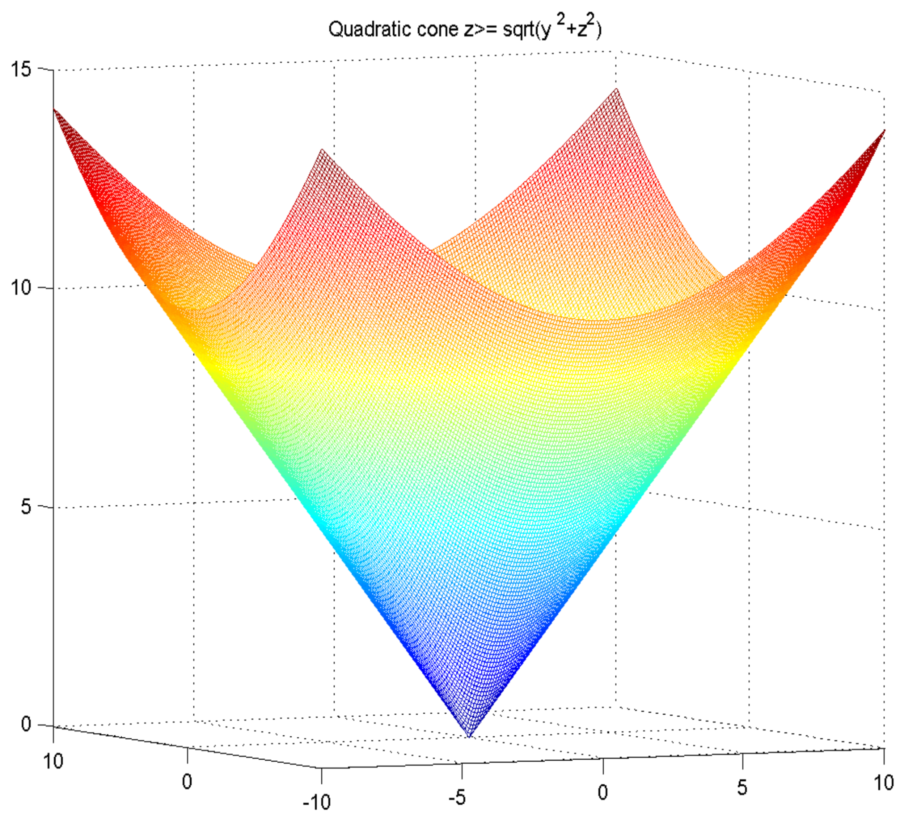
$$\begin{aligned} \min \quad & f^T x \\ s.t. \quad & Ax \le b \\ & x \in C, \quad x \in \left[l^x, u^x \right] \end{aligned}$$

where C is a second order cone (dim=k):

$$C_k = \left\{ \begin{bmatrix} x_{1:(k-1)} \\ x_k \end{bmatrix} : \ \left\| x_{1:(k-1)} \right\|_2 \le x_k \right\}$$

Example (quadratic cone)

Quadractic cone C sometimes also called Lorentz cone (or ice cream cone)



Trivial Quadratic Cone:

$$z \geq \sqrt{x^2 + y^2}$$

Second Order Cone Programs

$$\begin{aligned} \min \quad & f^T x \\ s.t. \quad & \|C_i x + d_i\| \le a_i^T x + b_i, \qquad i = 1,\dots, N \end{aligned}$$

$$x \in \Re^n \quad f, a_i \in \Re^n \quad C_i \in \Re^{(k_i - 1)\times n} \quad d_i \in \Re^{k_i - 1} \quad b_i \in \Re$$

Equivalent to conic program

- Linear constraints: cone dimension k=1
- Cone constraints: change of variables

$$\text{(vector) } y = C_i x + d_i, \qquad z = a_i^T x + b_i$$

Types of Cones

- Quadratic Cone

$$x_i \geq \sqrt{\sum_{j \neq i} x_i^2}$$

- Rotated Quadratic Cone

$$2x_i x_j \geq \sum_{k \neq i, k \neq j} x_k^2$$

Sometimes preferable for modeling quadratic inequalities

Rotated Quadratic Cone

- Show equivalence to quadratic cone:

$$2xy \geq \|z\|_2^2$$

Rotated quadratic cone

$$xy \geq \sum_i z_i^2 - xy \quad \Big| \quad + x^2 + y^2$$

$$(x+y) \geq \left\| \begin{bmatrix} z \\ (x-y) \end{bmatrix} \right\|$$

Quadratic cone

General Transformation

- If $x \in C^r \Leftrightarrow Ax \in C^q$ where

- C^r rotated quadratic cone
- C^q quadratic cone

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & \cdots & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Conic Constraints in GAMS

In the GAMS modeling language:

- Conic constraints denoted by `=C=`
- Conic programs result in “linear programs” in GAMS

Quadratic Cone in GAMS

```
Set s     /s1,s2,…,sn/;
Set t(s) /    s2,…,sn/;
Variable x(s);
```

Conic “LP” formulation

```
x(‘s1’) =C= sum(t(s), x(s));
```

Equivalent NLP formulation

```
x(‘s1’) =G= sqrt[sum(t(s),sqr(x(s)))];
```

→ Note: Summation on right hand side

Rotated Quadratic Cone

```
Set s      /s1,s2,s3,…,sn/;
Set t(s) /       s3,…,sn/;
Variable x(s);
```

Conic “LP” formulation

```
x(‘s1’)+x(‘s2’) =C= sum(t(s), x(s));
```

Equivalent NLP formulation

```
2*x(‘s1’)*x(‘s2’)=G= sum[t(s),sqr(x(s))];
```

→ Note: Summation on right hand side

Example: Complex L1 Norm

Complex L1 Norm Minimization

- Minimize $||Ax-b||_1$ where A,x,b are complex valued
- We can write this as

$$\min \left\| \begin{bmatrix} \mathrm{Re}(A) & -\mathrm{Im}(A) \\ \mathrm{Im}(A) & \mathrm{Re}(A) \end{bmatrix} \begin{pmatrix} \mathrm{Re}(x) \\ \mathrm{Im}(x) \end{pmatrix} - \begin{pmatrix} \mathrm{Re}(b) \\ \mathrm{Im}(b) \end{pmatrix} \right\|_2$$

GAMS

Example (Continued)

Complex L1 Norm Minimization

$$\min \quad \sum_i t(i)$$

$$\text{s.t.} \quad \begin{pmatrix} z_{re}(i) \\ z_{im}(i) \end{pmatrix} = \begin{bmatrix} \mathrm{Re}(A) & -\mathrm{Im}(A) \\ \mathrm{Im}(A) & \mathrm{Re}(A) \end{bmatrix} \begin{pmatrix} \mathrm{Re}(x) \\ \mathrm{Im}(x) \end{pmatrix} - \begin{pmatrix} \mathrm{Re}(b) \\ \mathrm{Im}(b) \end{pmatrix}$$

$$t(i) \quad \geq \quad \sqrt{z_{re}^2(i) + z_{im}^2(i)} \quad \text{(quadratic cone)}$$

GAMS Model (Quadratic Cone)

```
Objective..    obj         =E=   sum(i, t(i));

reseq_re(i).. res_re(i) =E=    sum(j, A_re(i,j)*x_re(j))
                              - sum(j,A_im(i,j)*x_im(j))
                              - b_re(i);

reseq_im(i).. res_im(i) =E=    …

coneeq(i)..   t(i)        =C=   res_re(i) + res_im(i);

Model conemodel /objective, reseq_re, reseq_im, coneeq/;

Solve conemodel using lp minimizing obj;
```

Portfolio Optimization

$$\min \quad \sum_{j,j'} x_j \sigma_{j,j'} x_{j'} = \alpha \|\underline{Dx}\|_2^2$$

$$s.t. \quad \sum_j x_j = 1, \qquad x_i \geq 0$$

$$\sum_j p_j x_j \geq r_{\min}$$

Objective is to minimize variance (risk), subject to an expected return

$\sigma_{j,j'}$ = covariance

♋=1 / (numdays-1)

x_j = % of investment in stock j

p_j = price change (return) for stock j

r_{min} = minimum expected return

$D_{j,d}$ = Deviation per day d of stock j wrt to mean return

Portfolio Optimization (Cont.)

Can rewrite: $min \;\; \alpha \|Dx\|_2^2$

By introducing intermediate variables p,q, and *w*:

$$\begin{aligned} \text{minimize} \quad & \alpha\, 2r \\ \text{subject to} \quad & w(d) = \sum_j D(j,d)x(j) \\ & q \qquad = 1 \\ & 2qr \qquad \geq \sum_d w(d)^2 \end{aligned}$$

GAMS Model (Rotated Cone)

```
Objective..    obj        =E=   a*(2*r);

Budget..       sum(j, x(j)) =E= 1;

Return..       Sum(j, p(j)*x(j)) =G= rmin;

Wcone(days).. w(d)    = sum(j, D(j,d)*x(j);

cone_eq1..     q =E= 1;

cone_eq2..     q + r   =C= sum(d, w(d));

Model conemodel  / all /;

Solve conemodel using lp minimizing obj;
```

GAMS/MOSEK

Solving Conic Models in GAMS:

- Newest addition is MOSEK
  - LP (simplex or interior point)
  - MIP (branch and bound)
  - Conic Programs (conic interior point):
  - Convex NLP
- Solver CONOPT also accepts conic constraints

Numerical Examples

DIMACS Challenge Models (SDP and SOCP)

- Chose subset of models from DIMACS (only SOCP models)

- SOCP models:
  - Sum of norms
  - Antenna array weight design
  - Scheduling problems

→ MOSEK solves all SOCP models and is the most efficient

Numerical Examples

DIMACS Benchmarks by Hans Mittelmann

- Solvers: MOSEK 2.5.1 (ext MPS), LOQO 6.03, SDPT3 3.01, SeDuMi 1.05R4

- 18 SOCP problems
  - In SeDuMi (MATLAB) format
  - MOSEK: extended QPS format (based on MPS)

DIMACS Results (H. Mittelmann)

Problem	LOQO	MOSEK	SDPT3	SeDuMi
Nb	11	3	11	9
Nb_L1	9	3	20	11
Nb_L2	16	8	18	24
Nb_L2_Bessel	7	2	11	12
Nql30	11	2	4	4
Nql60	151	9	14	9
Nql180	MM	182	232	459
Qssp30	15	2	6	3
Qssp60	221	9	29	18
Qssp180	MM	355	504	780
Sched_50_50_orig	5	1	6	6
Sched_100_50_orig	22	2	17	13
Sched_100_100_orig	94	4	28	29
Sched_200_100_orig	409	10	95	138
Sched_50_50_orig	7	1	6	5
Sched_100_50_orig	28	2	13	9
Sched_100_100_orig	107	5	22	32
Sched_200_100_orig	445	10	68	75

MM: memory problems

DIMACS Results (Cont.)

Use Performance Profiles (Dolan and Moré, 2002) to visualize results:

- Cumulative distribution function for a performance metric
- Performance metric: ratio τ of current solver time over best time of all solvers for "success"
- Intuitively: probability of success if given τ times fastest time (τ=ratio)

Profiles (Data: H. Mittelmann)

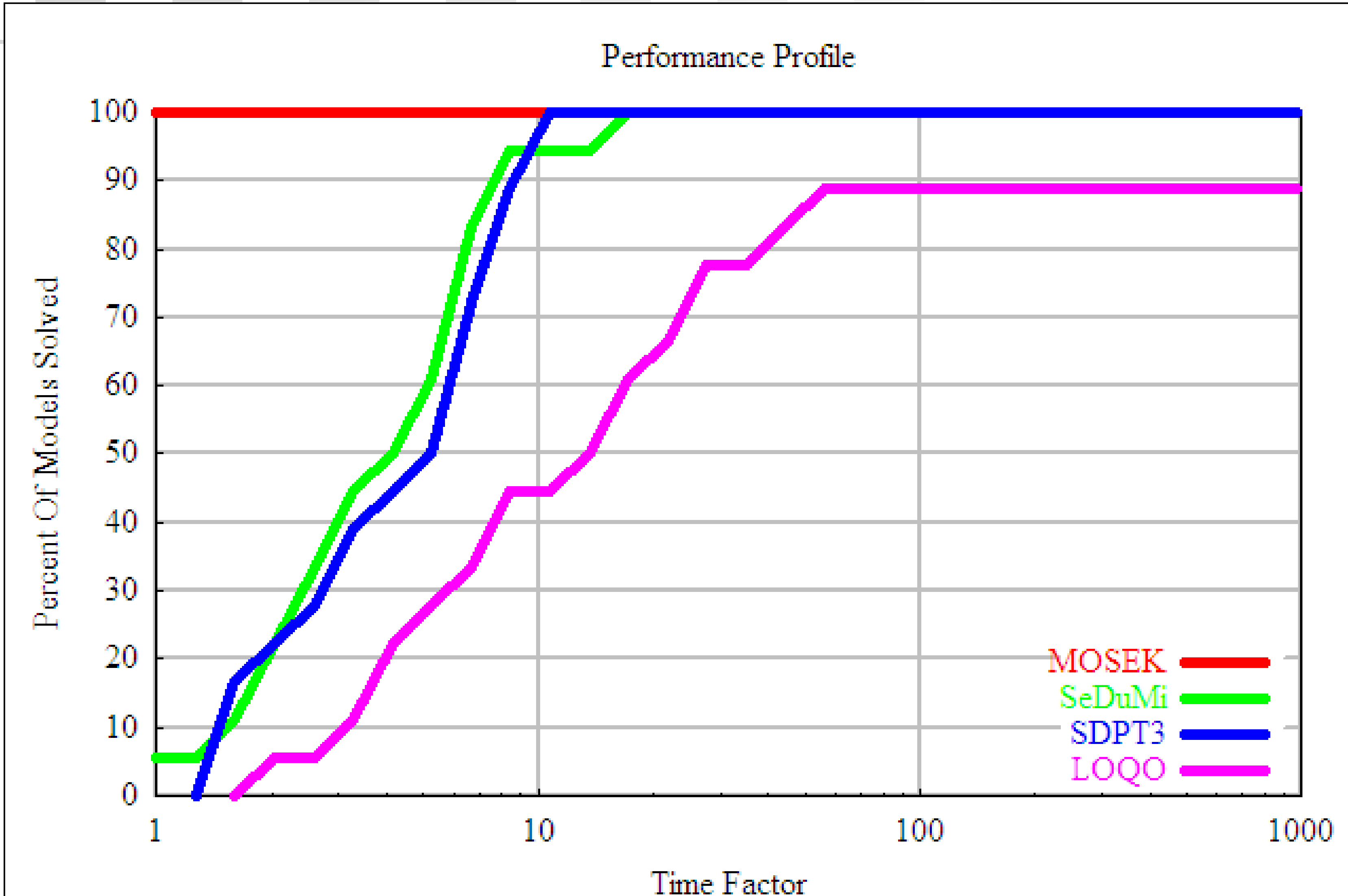
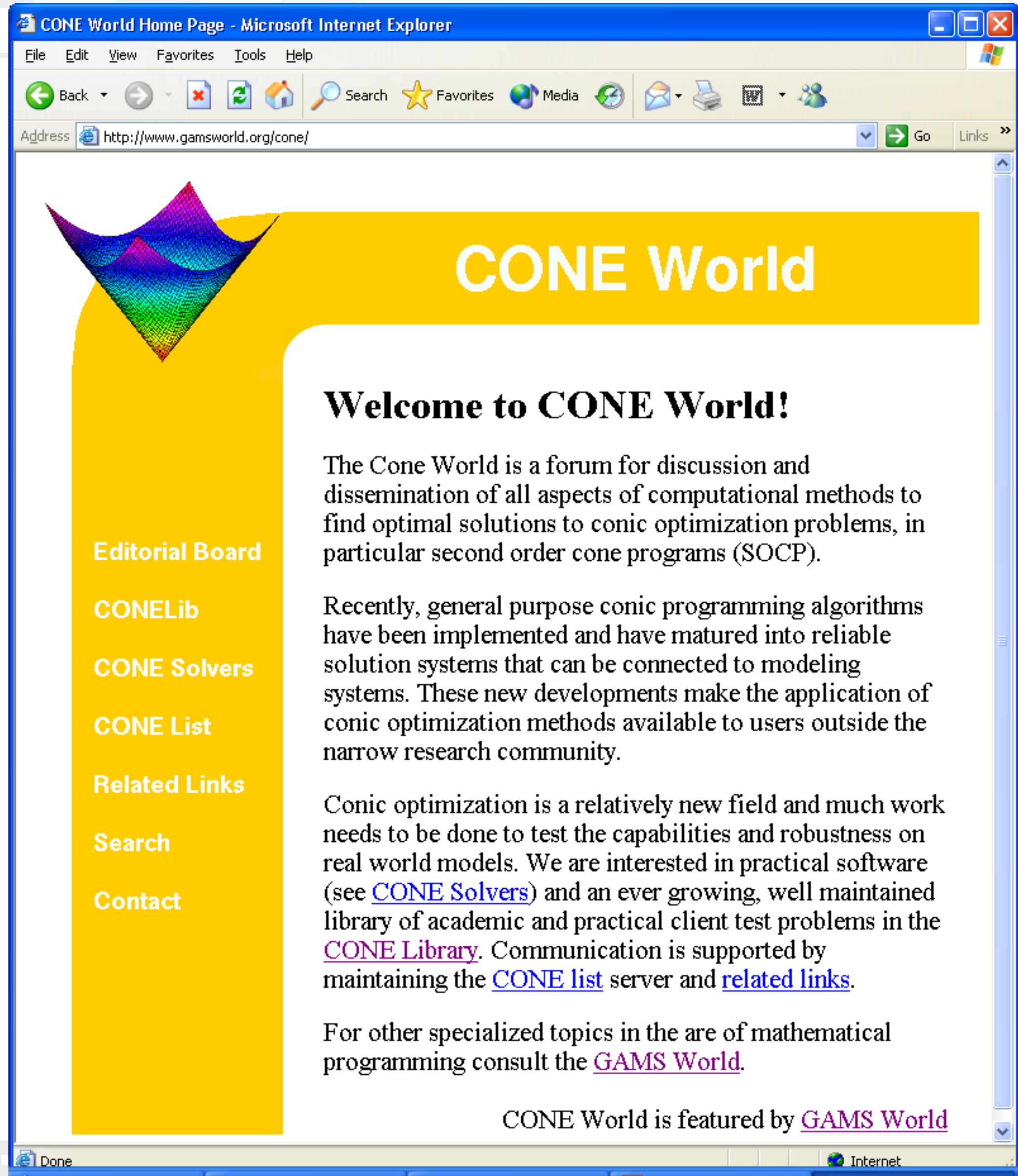


Figure 1

CONE World

- An online forum for discussion and information on cone programming
- CONELib library of models
  - GAMS cone format
  - NLP formulation
- Conic programming solvers
- Links and lists



ConeLib - Conic Models in GAMS

CONELib Model Statistics - Microsoft Internet Explorer

http://www.gamsworld.org/cone/conelib/conestat.htm

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CONELib Model Statistics

Total number of models: 28

Conic Models listed as LP in GAMS. Access the CONELIB in NLP format, converted using GAMS/CONVERT.

Name	Type	#Eqns	#ConeEqns	#Vars	#NZ	Bestknown	at Point
emfl_socp_200_25	LP	16876	5625	16926	41472	46.8675	p1
emfl_socp_200_100	LP	90001	30000	90201	236172		
emfl_socp_500_100	LP	180001	60000	180201	447957		
emfl_socp_250_225	LP	320626	106875	321076	877084		
fir_socp_10	LP	307	1	10	1711	-1.7382	p1
fir_socp_20	LP	307	1	15	3231	1.0001	p1
fir_socp_40	LP	307	1	25	6271		
fir_socp_80	LP	307	1	45	12351		
nb	LP	917	793	2384	193901		
nb_l1	LP	1709	793	3177	195487	-13.0123	p1
nb_l2	LP	963	839	4196	407317	-0.0507	p1
nb_l2_bessel	LP	963	839	2642	212295	-0.1026	p1
qp7	LP	34	1	82	1585	0.0008	p1
qp7_50_100	LP	34	1	82	1585	0.0018	p1
qp7_100_100	LP	34	1	82	1585	0.0043	p1
qp7_100_170	LP	34	1	82	1585	0.0034	p1
sched_50_50_orig	LP	2530	2	4980	27967	26672.9916	p1
sched_50_50_scaled	LP	2528	1	4978	30462	7.8520	p1
sched_100_50_orig	LP	4847	2	9747	60037	181889.9181	p1
sched_100_50_scaled	LP	4845	1	9745	65032	67.1651	p1
sched_100_100_orig	LP	8341	2	18241	113142		
sched_100_100_scaled	LP	8339	1	18239	123137	27.3308	p1
sched_200_100_orig	LP	18090	2	37890	278392		
sched_200_100_scaled	LP	18088	1	37888	298387	51.8120	p1

ConeLib - Conic Models in GAMS

nb_l1.gms - Microsoft Internet Explorer

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Address http://www.gamsworld.org/cone/conelib/nb_l1.htm

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nb_l1.gms:

References:

- Pataki, G, and Schmieta, S, The DIMACS library of semidefinite-quadratic-linear programs. Tech. rep., Computational Optimization Research Center, Columbia University, 2002.
- Pataki, G, and Schmieta, S, The DIMACS library of semidefinite-quadratic-linear programs, online at http://dimacs.rutgers.edu/Challenges/Seventh/Instances/.
- Original source: DIMACS Challenge SOCP (extended MPS format) from ftp://plato.asu.edu/pub/socp/

Point: p1

```
*  LP written by GAMS Convert at 10/14/03 16:40:08
*
*  Equation counts
*     Total        E        G        L        N        X        C
*      1709      916        0        0        0        0      793
*
*  Variable counts
*                  x        b        i      s1s      s2s       sc       si
*     Total     cont   binary  integer     sos1     sos2    scont     sint
*      3177     3177        0        0        0        0        0        0
*  FX     0        0        0        0        0        0        0        0
*
*  Nonzero counts
*     Total    const       NL      DLL
*    195487   195487        0        0
*
*  Solve m using LP minimizing objvar;
```

Internet

Cone Versus NLP Formulation

- Modeling conic constraints can be tedious
  - NLP form of quadratic cone is more natural
  - Rotated quadratic cone for QP is cumbersome

BUT

- Potentially significant computational advantages

- Compare cone vs. NLP formulation on CONELib (currently 28 models)
  - NLP formulation: substitute into conic constraint (converted using CONVERT utility in GAMS)
  - In practice "smarter" NLP formulations may exist depending on model

Cone Vs. NLP (Efficiency)

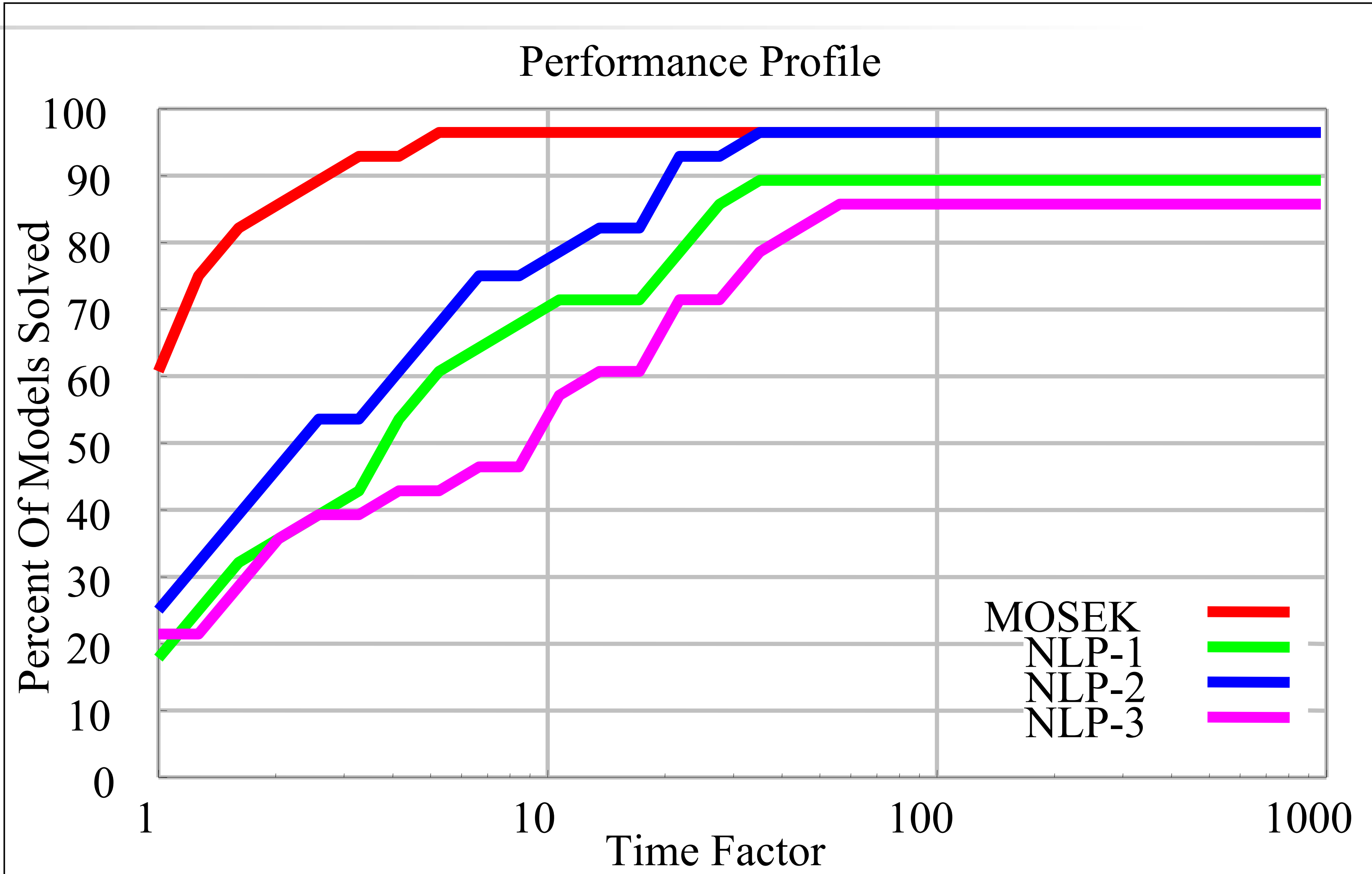

Figure 2

Reproducibility of Results

- Reproducibility of results is key to validity in scientific disciplines (but sometimes neglected)
  - Open data source (models, solvers, solver options)
  - Should be easily reproducible

- As part of the GAMS World, we provide downloadable scripts to reproduce results

- Currently only for NLP/Cone models (Global World)
  - See www.gamsworld.org/global/reproduce

GAMS

Reproducibility of Results

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GLOBAL Reproducibility of Results

Key to validity of scientific results is the reproducibility of numerical experiments. In this section we show how numerical results of papers can be reproduced, giving step-by-step instructions on reproducibility of results on local machines. We provide links to the models used, as well as scripts to run the actual experiments.

Papers and Presentations With Reproducible Numerical Results

- Pinter, J (2003), GAMS/LGO Nonlinear Solver Suite: Key Features, Usage, and Numerical Results. Submitted.
  - Instructions to reproduce the results
  - Results: Figure 1 (see the performance profile plot)
  - Results: Figure 2 (see the performance profile plot)
- Pruessner, A, Bussieck, M., Dirkse, S., and Meeraus, A. (2003), Conic Programming in GAMS, INFORMS Annual Meeting, Atlanta, October 19-22, Session on "Optimization Software - The State of the Art".
  - Instructions to reproduce the results
  - Results: Figure 2 (see the performance profile plot)

Internet

Reproducibility of Results

GLOBAL World - Links - Microsoft Internet Explorer

Address: http://www.gamsworld.org/global/reproduce/conic_informs2003.htm

GLOBAL Reproducibility of Results

This section gives instructions on reproducing the numerical results in the presentation

- **Pruessner, A, Bussieck, M., Dirkse, S., and Meeraus, A. (2003), Conic Programming in GAMS, INFORMS Annual Meeting, Atlanta, October 19-22, Session on "Optimization Software - The State of the Art".**

Note: The test set consists only of the 12 DIMACS SOCP problems converted to GAMS format. The solutions returned by MOSEK are all the optimal solutions, whereas the NLP solvers give different solutions. Thus, only the "efficiency" profile plot makes sense.

Version/Platform:

- GAMS 21.2 (Linux).

Preliminaries

- Download the CONELib library of models: conelib.zip
- Download the GAMS script to run the experiments
  - conetest_fig.gms
- Download the termination routines described in the section on Program Termination These tools are necessary to independently ensure that solvers are allocated a set resource solver time.
  - The GAMS routine schulz.gms
  - PSList (Download and place into your GAMS system directory)
  - PSKill (Download and place into your GAMS system directory)

  The tools PSKill and PSList are free from but proprietary from Sysinternals.

Reproduce Results in Figure 1

- Run the schulz.gms termination routine. From the command line

```
>> gams schulz.gms --resseq=65:70:75
```

On Windows: Add the flag `--watch=gms*`

- Run the GAMS script gotest_fig1.gms. From the command line

```
>> gams conetest_fig.gms
```

Conclusions

- Addition of conic programming capability within GAMS
- MOSEK as state-of-the-art conic programming solver
- Although modeling still cumbersome, significant computational advantages using cones
- CONELib: growing collection of conic programming models
- Presentation will be available under
  `http://www.gams.com/presentations`

References

- Lobo, M.S., Vandenberghe, L., Boyd, S. and Lebret, H. *Applications of Second Order Cone Programming*, Linear Algebra and its Applications, 284:193-228, November 1998.
- MOSEK Optimization Tools Help Desk, Version 2, online at www.mosek.com/documentation, 2003.
- H.D. Mittelmann, *An independent benchmarking of SDP and SOCP solvers*, Math Program., Series B, 2002 (appeared electronically).
- E. D. Dolan and J. J. More, *Benchmarking optimization software with performance profiles*, Math. Programming, 91 (2), 201-213, 2002.