Mixed Integer Nonlinear Programs Theory, Algorithms & Applications

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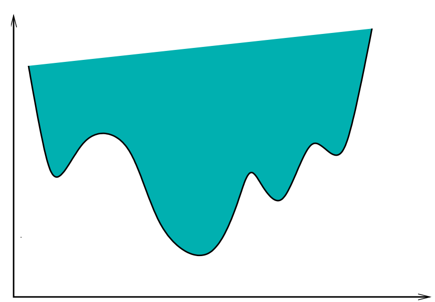
Problem Formulation

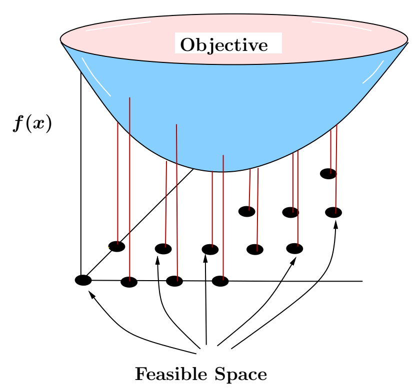
(P)	min	$f(x, y)$	Objective Function
	s.t.	$g(x, y) \leq 0$	Constraints
		$x \in \mathbb{Z}^p$	Integrality Restrictions
		$y \in \mathbb{R}^n$	Continuous Variables

Challenges:

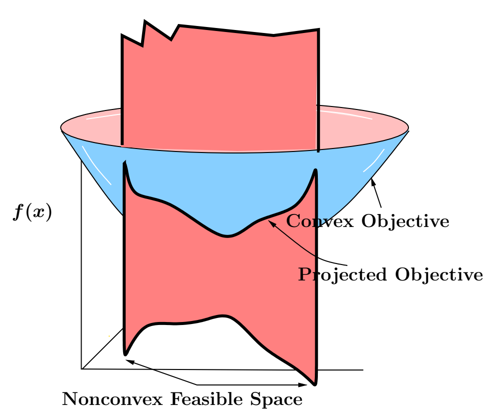
- Multimodal Objective



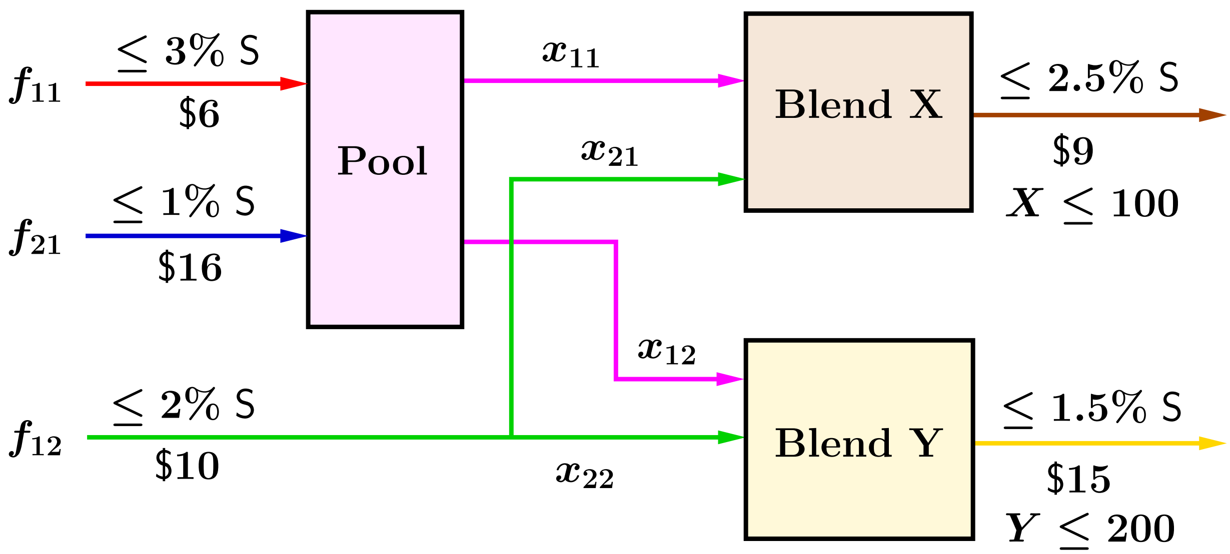
- Integrality



- Nonconvex Constraints



The Pooling Problem



$$\min \quad \overbrace{6f_{11} + 16f_{21} + 10f_{12}}^{\text{cost}} - \overbrace{9(x_{11} + x_{21})}^{X\text{-revenue}} - \overbrace{15(x_{12} + x_{22})}^{Y\text{-revenue}}$$

s.t. $q = \dfrac{3f_{11} + f_{21}}{x_{11} + x_{12}}$ **Concentration Balance**

$$f_{11} + f_{21} = x_{11} + x_{12}$$
$$f_{12} = x_{21} + x_{22}$$

Mass balance

$$\frac{qx_{11} + 2x_{21}}{x_{11} + x_{21}} \leq 2.5$$
$$\frac{qx_{12} + 2x_{22}}{x_{12} + x_{22}} \leq 1.5$$

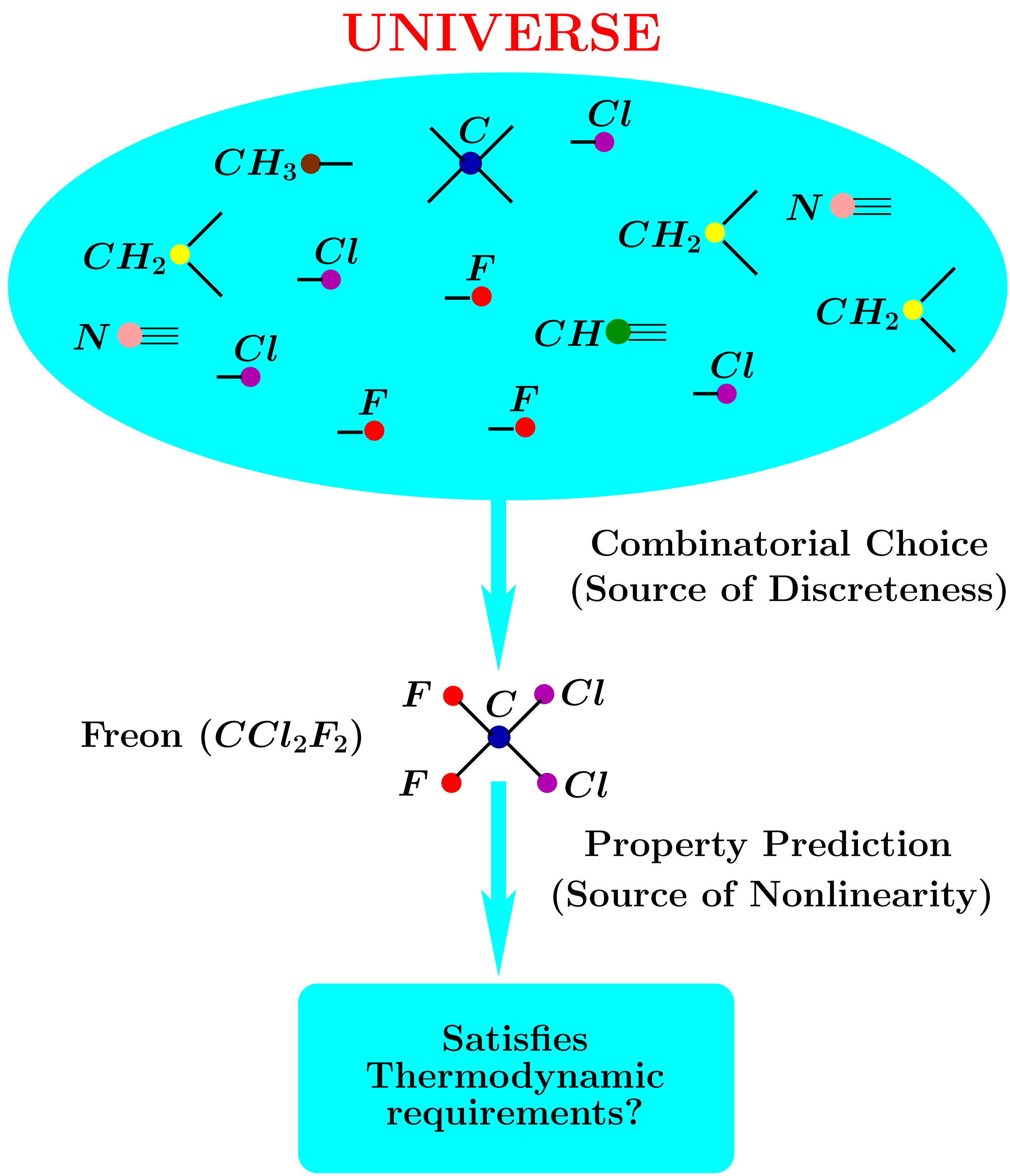
Quality Requirements

$$x_{11} + x_{21} \leq 100$$
$$x_{12} + x_{22} \leq 200$$

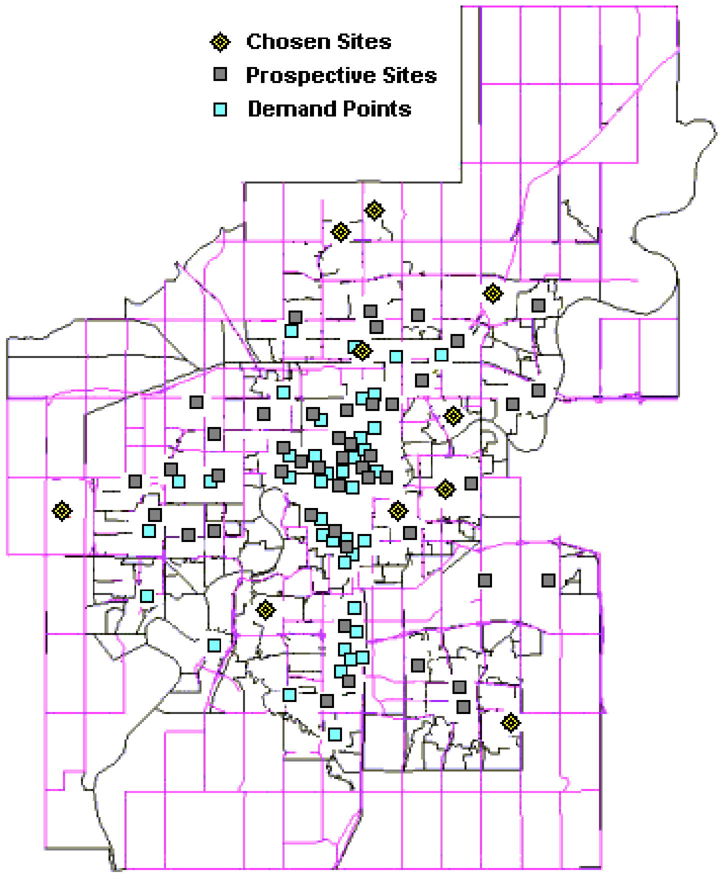
Demands

Molecular Design Problems

(Example: Refrigerant Design)



Discrete Location Problems



Restaurants in Edmonton

$$\textbf{p-choice Utility} = \sum_{\text{location: } j} w_j \sum_{\text{customer: } i} \underbrace{\frac{U_{ij} x_j}{\sum_k U_{ik} x_k}}_{\text{Utility Ratio}} d_i$$

U_{ij}	**Utility of location j for customer i**
x_j	**decision variable for locating facility j**
d_i	**demand by customer i**
w_j	**preferential weight of site j**

Motivation for an Algorithm

- **Applications:**
  - Combinatorial Chemistry and Molecular Biology
  - Utility Maximization (Risk Minimization)
  - Parameter Estimation (Design of Experiments)
  - Engineering Design (Process Synthesis, CAD)
  - Layout Design (Area Restrictions)
- **Theoretical Challenges**
- **Important Subclasses:**
  - Mixed Integer Linear Programming
  - Continuous Nonlinear Programming
  - Nonlinear 0-1 Programming
- **Search for Globality (Deterministic):**
  - Either optimality assured or a gap estimate
  - Rigorous termination criteria
  - Verifying local optimality is $\mathcal{NP}$-hard (Murty and Kabadi 1987)

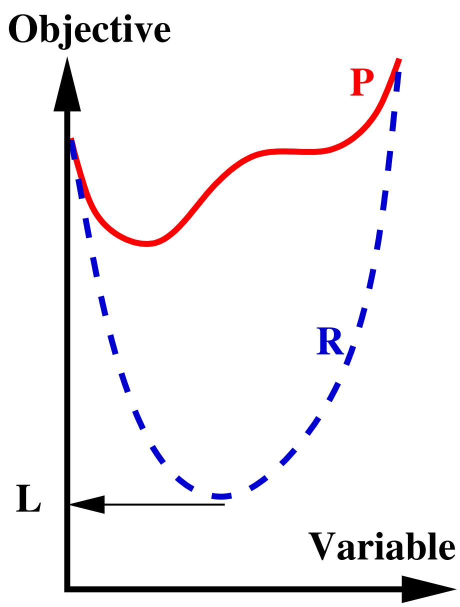
Deterministic Approaches

- **Branch and Bound**
  - **Bound the problem over successively refined partitions**
- **Convexification**
  - **Outer-approximate with increasingly tighter convex programs**
- **Decomposition**
  - **Project out some variables (move them to a subproblem)**

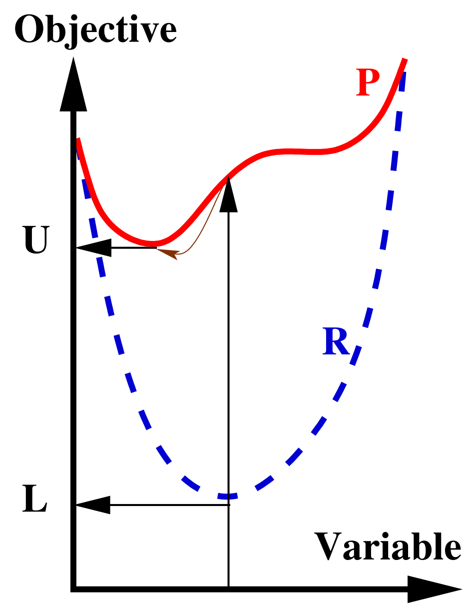
Our approach

Branch and Bound + Convexification

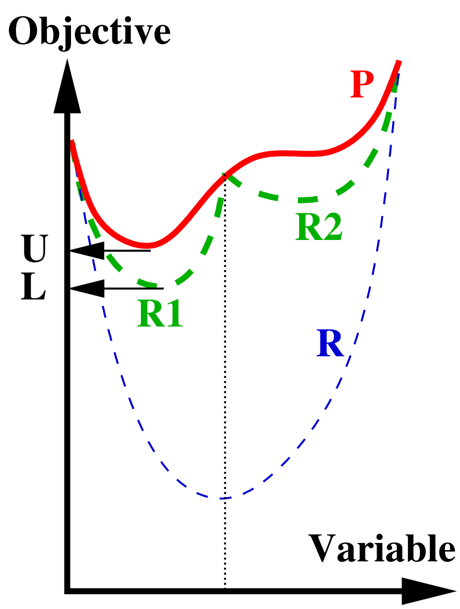
Branch and Bound Algorithm



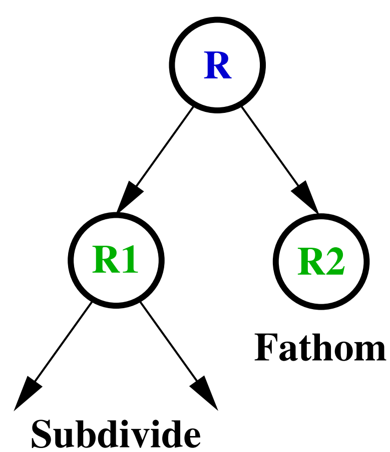
a. Lower Bounding



b. Upper Bounding

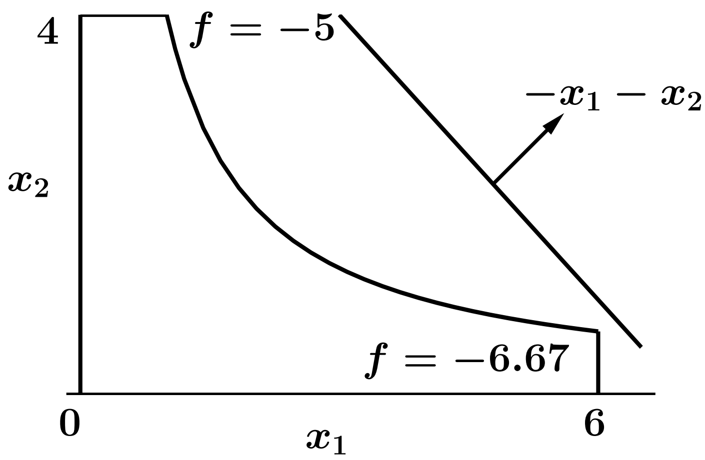


c. Domain Subdivision



d. Search Tree

Illustrative Example



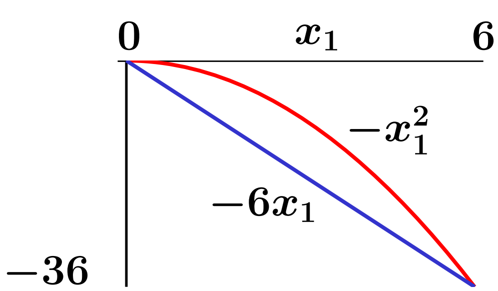
$$\begin{aligned} \min\ & -x_1 - x_2 \\ \text{s.t.}\ & x_1 x_2 \leq 4 \\ & 0 \leq x_1 \leq 6 \\ & 0 \leq x_2 \leq 4 \end{aligned}$$

Separable Reformulation : $x_1 x_2 = \frac{1}{2}\left\{(x_1 + x_2)^2 - x_1^2 - x_2^2\right\}$

Reformulation

$$\begin{aligned} \min\ & -x_1 - x_2 \\ \text{s.t.}\ & (x_1 + x_2)^2 - x_1^2 - x_2^2 \leq 8 \\ & 0 \leq x_1 \leq 6 \\ & 0 \leq x_2 \leq 4 \end{aligned}$$

Relaxation (not best)



$$\begin{aligned} \min\ & -x_1 - x_2 \\ \text{s.t.}\ & (x_1 + x_2)^2 - 6x_1 - 4x_2 \leq 8 \\ & 0 \leq x_1 \leq 6 \\ & 0 \leq x_2 \leq 4 \end{aligned}$$

Relaxation Solution: $x_1 = 6$, $x_2 = 0.89$, $L = -6.89$

Local Search (MINOS): $x_1 = 6$, $x_2 = 0.67$, $L = -6.67$

Relaxation Solution provides good starting point for local search

Challenges

Standard Global Optimization Converges Slowly

Algorithmic Issues:

- Tight Relaxations
- Domain Reduction
- Finiteness Issues

Implementation Issues:

- Existing NLP solvers often unreliable
- General purpose (automatic) software
- Enable solution of industrially relevant problems

Factorable Functions
(McCormick, 1976)

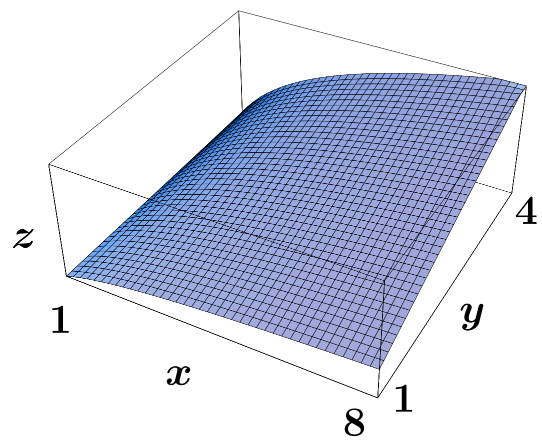
Definition: The factorable functions are recursive compositions of sums and products of functions of single variables.

Example: $f(x, y) = \sqrt{\exp(xy + z \ln w) z^3}$

$$\overbrace{\underbrace{\Big(\overbrace{\exp(\underbrace{\underbrace{xy}_{x_1} + \overbrace{z \underbrace{\ln w}_{x_2}}^{x_3}}_{x_4})}^{x_5}\ \overbrace{z^3}^{x_6}\Big)}_{x^7}{}^{0.5}}^{f}$$

$$x_1 = xy$$
$$x_2 = \ln(w)$$
$$x_3 = z x_2$$
$$x_4 = x_1 + x_3$$
$$x_5 = \exp(x_4)$$
$$x_6 = z^3$$
$$x_7 = x_5 x_6$$
$$f = \sqrt{x_7}$$

Challenges Via Example: $y \log(x)$

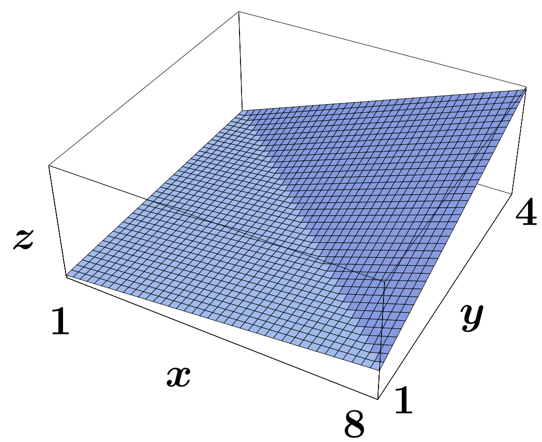


$$z \geq y \log(x)$$
$$1 \leq x \leq 8$$
$$1 \leq y \leq 4$$

$\Rightarrow$

$$z \geq y l_x$$
$$l_x \geq \log(x)$$
$$1 \leq x \leq 8$$
$$1 \leq y \leq 4$$

Convex Relaxation (McCormick style)



$$\bar{z} \geq \log(8)y + 4\bar{l_r} - 4\log(8)$$
$$\bar{z} \geq \bar{l_r}$$
$$\bar{l_r} \geq \frac{\log(8)}{7}(x-1)$$

Q: In general do we get the convex envelope? NO

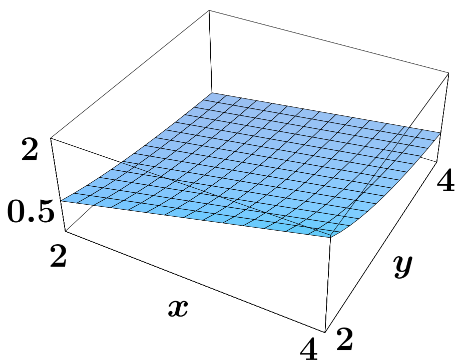
Q: In this case? YES (Convex Extensions Wed-9:00am)

Q: Simplification? $(\log 8/7) \max\{7y + 4x - 32, x - 1\}$

- Non-differentiability
- $\log(x)$ occurs elsewhere? (common subexpressions)

Q: What if $y = x$? (Note: $x \log x$ is convex)

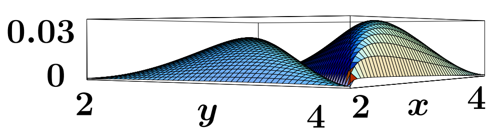
Ratio: Convex Extensions



Ratio: x/y

$$z \geq x/y$$
$$2 \leq x \leq 4$$
$$2 \leq y \leq 4$$

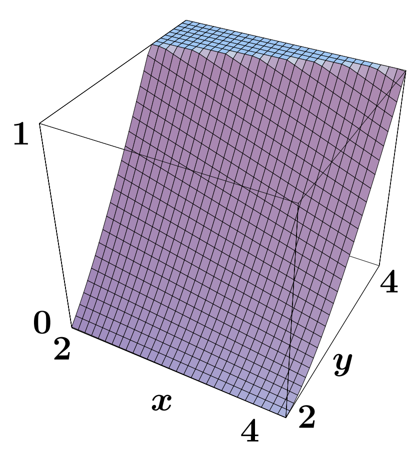
Convex Extensions yield Convex Envelope



x/y - Envelope

$$y_p z_p \geq x^L \left(\frac{x^U - x}{x^U - x^L}\right)^2$$
$$y_p(z - z_p) \leq y(z - z_p) - x^U \left(\frac{x - x^L}{x^U - x^L}\right)^2$$
$$y_p \geq \max\left\{y^L \frac{x^U - x}{x^U - x^L}, y - y^U \frac{x - x^L}{x^U - x^L}\right\}$$
$$y_p \leq \min\left\{y^U \frac{x^U - x}{x^U - x^L}, y - y^L \frac{x - x^L}{x^U - x^L}\right\}$$
$$z - z_p, z_p \geq 0$$

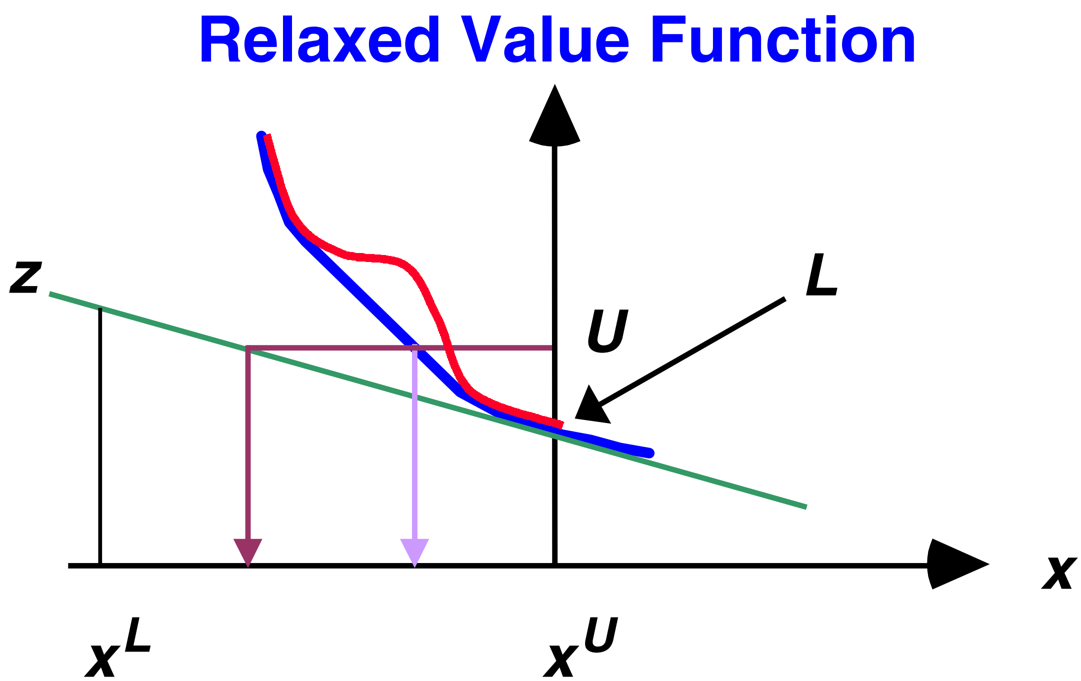
Factorable Relaxation is Weak



x/y - Factorable

$$z \geq -0.5 + x/4 + y/8$$
$$z \geq 2 + x/2 - y$$
$$2 \leq x \leq 4$$
$$2 \leq y \leq 4$$

RANGE REDUCTION

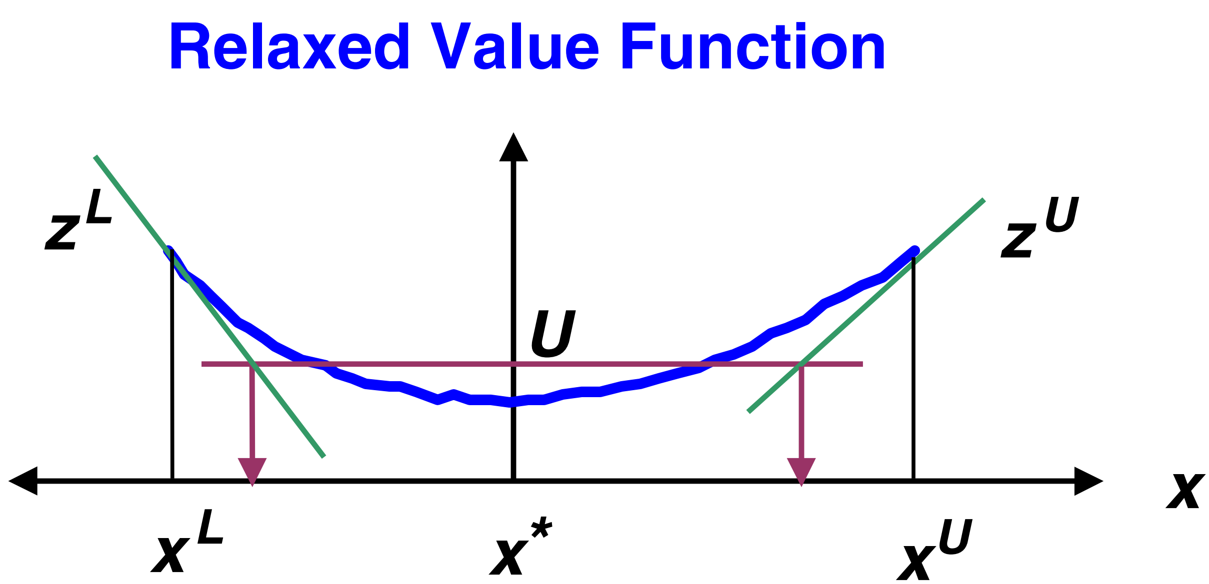


- **If a variable goes to its upper bound at the relaxed problem solution, this variable's lower bound can be improved**

PROBING

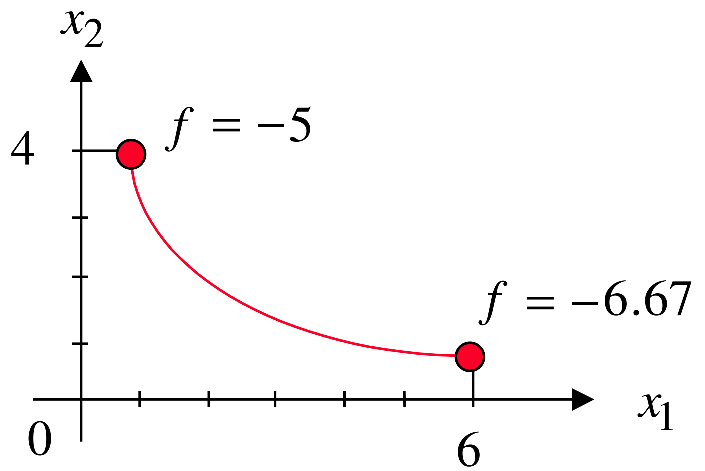
Q. What if a variable does not go to a bound?

A. Use probing: temporarily fix variable at a bound.



ILLUSTRATIVE EXAMPLE

$$\left.\begin{array}{ll} \min & f = -x_1 - x_2 \\ s.t. & x_1 x_2 \le 4 \\ & 0 \le x_1 \le 6 \\ & 0 \le x_2 \le 4 \end{array}\right\} (P)$$



Reformulation:

$$\begin{array}{ll} \min & f = -x_1 - x_2 \\ s.t. & x_3^2 - x_1^2 - x_2^2 \le 8 \\ & x_3 = x_1 + x_2 \\ & 0 \le x_1 \le 6 \\ & 0 \le x_2 \le 4 \\ & 0 \le x_3 \le 10 \end{array}$$

Relaxation:

$$\left.\begin{array}{ll} \min & -x_1 - x_2 \\ s.t. & x_3^2 - 6x_1 - 4x_2 \le 8 \\ & x_3 = x_1 + x_2 \\ & 0 \le x_1 \le 6 \\ & 0 \le x_2 \le 4 \\ & 0 \le x_3 \le 10 \end{array}\right\} (R)$$

Solution of R : $x_1 = 6, x_2 = 0.89, L = -6.89, \lambda_1 = 0.2$

Local Solution of P with MINOS : $U = -6.67$

Range reduction : $x_1^L \leftarrow x_1^U - (U - L)/\lambda_1 = 4.86$

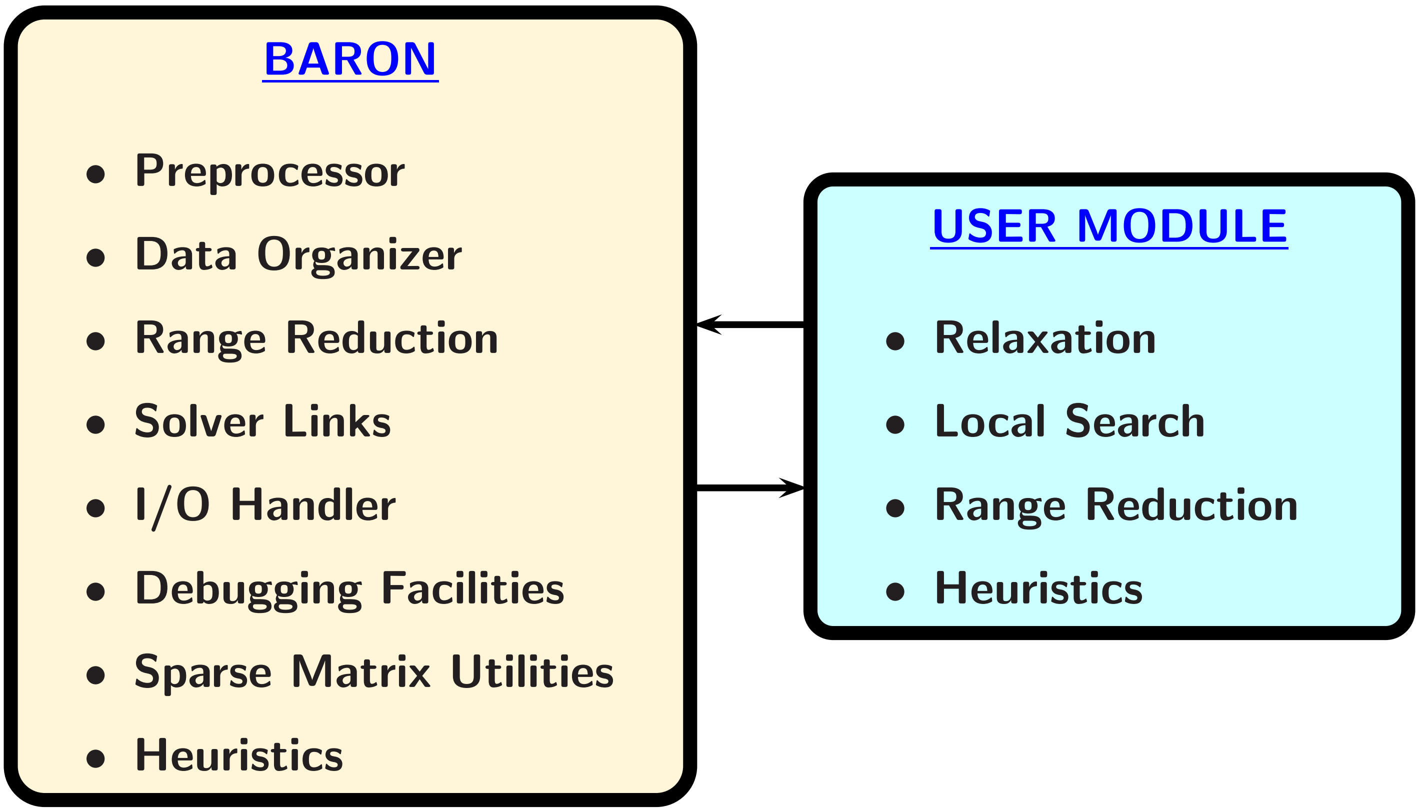
Probing (Solve R with $x_2 \le 0$) : $L = -6, \lambda_2 = 1 \Rightarrow x_2^L = 0.67$

Update R with $x_1 \ge 4.86, x_2 \ge 0.67$

Solution is : $L = -6.67$

∴ Proof of globality with NO Branching!!

Branch And Reduce Optimization Navigator



Core Module:

- expandable, application-independent

Application Modules:

- factorable, linear multiplicative, bilinear, concave, fixed-charge, integer, fractional programming
- solve convex relaxations using CPLEX, OSL, MINOS, SNOPT, SDPA

Factorable NLP Module

Formulation:

$$
\begin{array}{lll}
\min & f(x,y) & \\
 & a \le g_i(x,y) \le b & i = 1,\ldots,m \\
 & x_j^L \le x_j \le x_j^U & j = 1,\ldots,n \\
 & y_j^L \le y_j \le y_j^U & j = 1,\ldots,p \\
 & x_j \text{ real} & \\
 & y_j \text{ integer} &
\end{array}
$$

Factorable Functions:

f and g are recursive sums, products and ratios of the following terms:

- exponential, $\exp(x)$
- logarithmic, $\log(x)$
- monomial, x^a

Example: (Factorable Constraint)

$$1 \le \frac{x^2 y^{0.3} z}{p^2} + \exp\left(\frac{x^2 p}{y}\right) - xy \le 100$$

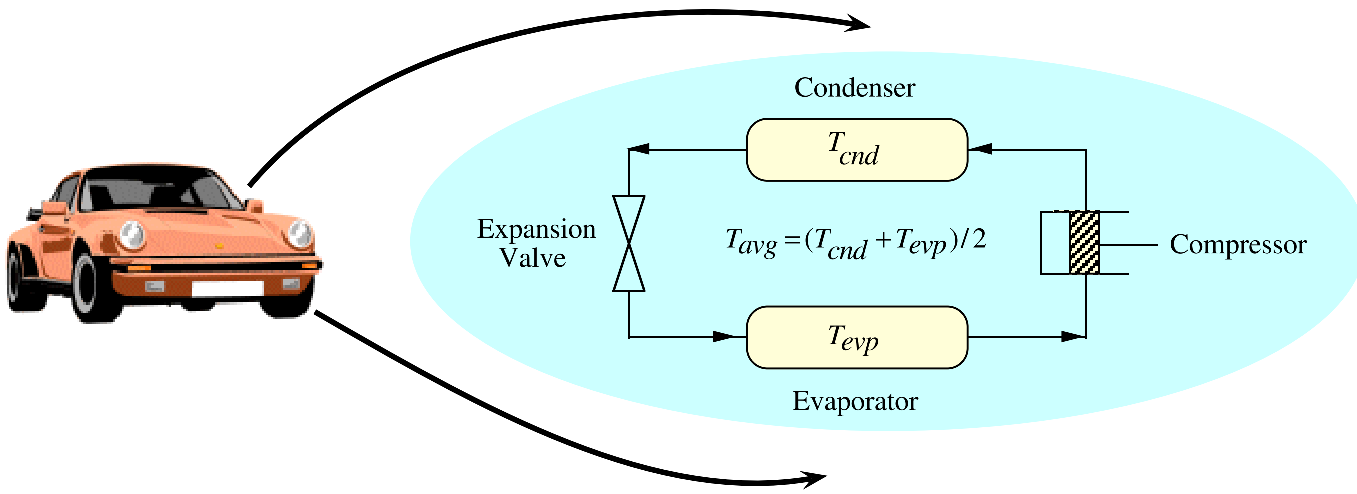
Pooling Problems in Literature

Algorithm	Foulds '92		Ben-Tal '94	GOP '96		BARON	
Computer*	CDC 4340			HP9000/730		RS6000/43P	
Linpack	> 3.5			49		59.9	
Tolerance*				**		10^{-6}	
Problem	N_{tot}	T_{tot}	N_{tot}	N_{tot}	T_{tot}	N_{tot}	T_{tot}
Haverly 1	5	0.7	3	12	0.22	3	0.08
Haverly 2			3	12	0.21	9	0.10
Haverly 3			3	14	0.26	3	0.11
Foulds 2	9	3				1	0.05
Foulds 3	1	10.5				1	1.38
Foulds 4	25	125				1	1.55
Foulds 5	125	163.6				1	0.43
Ben-Tal 4			25	7	0.95	3	0.08
Ben-Tal 5			283	41	5.80	1	2.27
Example 1						1869	77
Example 2						2087	146
Example 3						7369	1160
Example 4						157	9.5

* Blank indicates problem not reported or not solved

** 0.05% for Haverly 1, 2, 3, 0.05% for Ben-Tal 4 and 1% for Ben-Tal 5

Automotive Refrigerant Design (Joback and Stephanopoulos, 1984)



$$\max \quad \frac{\Delta H_{ve}}{C_{pla}}$$

$$\text{s.t.} \quad \Delta H_{ve} \geq 18.4$$

$$C_{pla} \leq 32.2$$

$$P_{vpe} \geq 1.4$$

Property Prediction Constraints

Structure Feasibility Constraints

Nonnegativity Constraints

Integrality Requirements

- High enthalpy of vaporization ΔH_{ve} reduces the amount of refrigerant
- Lower liquid heat capacity C_{pla} reduces amount of vapor generated in expansion valve

Functional Groups Considered

Acyclic Groups	Cyclic Groups	Halogen Groups	Oxygen Groups	Nitrogen Groups	Sulfur Groups
$-CH_3$	${}^{r}-CH_2-{}^{r}$	$-F$	$-OH$	$-NH_2$	$-SH$
$-CH_2-$	${}^{r}_{r}>CH-{}^{r}$	$-Cl$	$-O-$	$>NH$	$-S-$
$>CH-$	${}_{r}>CH-{}^{r}$	$-Br$	${}^{r}-O-{}^{r}$	${}^{r}_{r}>NH$	${}^{r}-S-{}^{r}$
$>C<$	${}^{r}_{r}>C<{}^{r}_{r}$	$-I$	$>CO$	$>N-$	
$=CH_2$	${}_{r}>C<{}^{r}_{r}$		${}^{r}_{r}>CO$	$=N-$	
$=CH-$	$>C<{}^{r}_{r}$		$-CHO$	${}^{r}=N-{}^{r}$	
$=C<$	${}^{r}=CH-{}^{r}$		$-COOH$	$-CN$	
$=C=$	${}^{r}=C<{}^{r}_{r}$		$-COO-$	$-NO_2$	
$\equiv CH$	${}^{r}=C<{}_{r}$		$=O$		
$\equiv C-$	$=C<{}^{r}_{r}$				

Number of Groups = 44

Maximum Selection Size = 15

Candidates = 39, 895, 566, 894, 524

Property Prediction Constraints

$$T_b = 198.2 + \sum_{i=1}^{N} n_i T_{bi}$$

$$T_c = \frac{T_b}{0.584 + 0.965 \sum_{i=1}^{N} n_i T_{ci} - \left(\sum_{i=1}^{N} n_i T_{ci}\right)^2}$$

$$P_c = \frac{1}{\left(0.113 + 0.0032 \sum_{i=1}^{N} n_i a_i - \sum_{i=1}^{N} n_i P_{ci}\right)^2}$$

$$C_{p0a} = \sum_{i=1}^{N} n_i C_{p0ai} - 37.93 + \left(\sum_{i=1}^{N} n_i C_{\mathrm{p0bi}} + 0.21\right) T_{\mathrm{avg}} + \left(\sum_{i=1}^{N} n_i C_{p0ci} - 3.91 \times 10^{-4}\right) T_{\mathrm{avg}}^2 + \left(\sum_{i=1}^{N} n_i C_{\mathrm{p0di}} + 2.06 \times 10^{-7}\right) T_{\mathrm{avg}}^3$$

$$T_{\mathrm{br}} = \frac{T_{\mathrm{b}}}{T_c}$$

$$T_{\mathrm{avgr}} = \frac{T_{\mathrm{avg}}}{T_c}$$

$$T_{\mathrm{cndr}} = \frac{T_{\mathrm{cnd}}}{T_c}$$

$$T_{\mathrm{evpr}} = \frac{T_{\mathrm{evp}}}{T_c}$$

$$\alpha = -5.97214 - \ln\left(\frac{P_c}{1.013}\right) + \frac{6.09648}{T_{\mathrm{br}}} + 1.28862 \ln(T_{\mathrm{br}}) - 0.169347 T_{\mathrm{br}}^6$$

$$\beta = 15.2518 - \frac{15.6875}{T_{\mathrm{br}}} - 13.4721 \ln(T_{\mathrm{br}}) + 0.43577 T_{\mathrm{br}}^6$$

$$\omega = \frac{\alpha}{\beta}$$

$$C_{\mathrm{pla}} = \frac{1}{4.1868} \left\{ C_{\mathrm{p0a}} + 8.314 \left[1.45 + \frac{0.45}{1 - T_{\mathrm{avgr}}} + 0.25\omega \left(17.11 + 25.2 \frac{(1 - T_{\mathrm{avgr}})^{1/3}}{T_{\mathrm{avgr}}} + \frac{1.742}{1 - T_{avgr}} \right) \right] \right\}$$

$$\Delta H_{\mathrm{vb}} = 15.3 + \sum_{i=1}^{N} n_i \Delta H_{\mathrm{vbi}}$$

$$\Delta H_{\mathrm{ve}} = \Delta H_{\mathrm{vb}} \left(\frac{1 - T_{\mathrm{evp}}/T_c}{1 - T_b/T_c} \right)^{0.38}$$

$$h = \frac{T_{br} \ln(P_c/1.013)}{1 - T_{br}}$$

$$G = 0.4835 + 0.4605h$$

$$k = \frac{h/G - (1 + T_{br})}{(3 + T_{br})(1 - T_{br})^2}$$

$$\ln P_{\mathrm{vpcr}} = \frac{-G}{T_{\mathrm{cndr}}} \left[1 - T_{\mathrm{cndr}}^2 + k(3 + T_{\mathrm{cndr}})(1 - T_{\mathrm{cndr}})^3 \right]$$

$$\ln P_{\mathrm{vper}} = \frac{-G}{T_{\mathrm{evpr}}} \left[1 - T_{\mathrm{evpr}}^2 + k(3 + T_{\mathrm{evpr}})(1 - T_{\mathrm{evpr}})^3 \right]$$

n_i integer

Structure Feasibility Constraints

$$\sum_{i=1}^{N} n_i \geq 2$$

$$Y_A \leq \sum_{i \in \mathcal{A}} n_i \leq N_{max} Y_A \|\mathcal{A}\|$$

$$Y_C \leq \sum_{i \in \mathcal{C}} n_i \leq N_{max} Y_C \|\mathcal{C}\|$$

$$Y_M \leq \sum_{i \in \mathcal{M}} n_i \leq N_{max} Y_M \|\mathcal{M}\|$$

$$Y_A + Y_C - 1 \leq Y_M \leq Y_A + Y_C$$

$$3Y_R \leq \sum_{i \in \mathcal{R}} n_i \leq N_{max} Y_R \|\mathcal{R}\|$$

$$\sum_{i=1}^{N} n_i b_i \geq 2\left(\sum_{i=1}^{N} n_i - 1\right);$$

$$\sum_{i=1}^{N} n_i b_i \leq \left(\sum_{i=1}^{N} n_i\right)\left(\sum_{i=1}^{N} n_i - 1\right)$$

$$\sum_{i \in \mathcal{SD}} n_i \geq 1 \text{ if } \sum_{i \in \mathcal{S}/\mathcal{D}} n_i \geq 1 \text{ and } \sum_{i \in \mathcal{D}/\mathcal{S}} n_i \geq 1$$

$$\sum_{i \in \mathcal{ST}} n_i \geq 1 \text{ if } \sum_{i \in \mathcal{S}/\mathcal{T}} n_i \geq 1 \text{ and } \sum_{i \in \mathcal{T}/\mathcal{S}} n_i \geq 1$$

$$\sum_{i \in \mathcal{SSR}} n_i \geq 1 \text{ if } \sum_{i \in \mathcal{S}/\mathcal{SR}} n_i \geq 1 \text{ and } \sum_{i \in \mathcal{SR}/\mathcal{S}} n_i \geq 1$$

$$\sum_{i \in \mathcal{SDR}} n_i \geq 1 \text{ if } \sum_{i \in \mathcal{S}/\mathcal{DR}} n_i \geq 1 \text{ and } \sum_{i \in \mathcal{DR}/\mathcal{S}} n_i \geq 1$$

$$\sum_{i \in \mathcal{DSR}} n_i \geq 1 \text{ if } \sum_{i \in \mathcal{D}/\mathcal{SR}} n_i \geq 1 \text{ and } \sum_{i \in \mathcal{SR}/\mathcal{D}} n_i \geq 1$$

$$\sum_{i \in \mathcal{B}} n_i = 2Z_B; \quad \sum_{i \in \mathcal{S}} n_i = 2Z_S; \quad \sum_{i \in \mathcal{SR}} n_i = 2Z_{SR}$$

$$\sum_{i \in \mathcal{D}} n_i = 2Z_D$$

$$\sum_{i \in \mathcal{DR}} n_i = 2Z_{DR}$$

$$\sum_{i \in \mathcal{T}} n_i = 2Z_T$$

$$\sum_{i} n_i(2 - b_i) = 2m$$

$$\sum_{i=1}^{N} n_i \geq n_j(b_j - 1) + 2 \quad j = 1, \ldots, N$$

$$\sum_{\substack{i \in \mathcal{O}, \\ \text{single-bonded}}} n_i \leq \sum_{i \in \mathcal{H}} n_i S_{ai} \text{ if } \sum_{i \in \mathcal{H}} n_i \neq 0$$

$$\sum_{\substack{i \in \mathcal{O}, \\ \text{double-bonded}}} n_i \leq \sum_{i \in \mathcal{H}} n_i D_{ai} \text{ if } \sum_{i \in \mathcal{H}} n_i \neq 0$$

$$\sum_{\substack{i \in \mathcal{O}, \\ \text{triple-bonded}}} n_i \leq \sum_{i \in \mathcal{H}} n_i T_{ai} \text{ if } \sum_{i \in \mathcal{H}} n_i \neq 0$$

$$\sum_{i \in \mathcal{H}} n_i(S_{ai} + D_{ai} + T_{ai}) - \sum_{i \in \mathcal{O}} n_i \leq 2\left(\sum_{i \in \mathcal{H}} n_i - 1\right)$$

Molecular Structures

	Molecular Structure	$\frac{\Delta H_{ve}}{C_{pla}}$
ClNO	$(Cl-)(-N=)(=O)$	1.5971
IF	$(I-)(-F)$	1.5297
		1.5219
O_3	$({}^{r} - O - {}^{r})_3$	1.4855
		1.3464
FNO	$(F-)(-N=)(=O)$	1.2880
		1.2767
		1.2594
CHClO	$(Cl-)(-CH=)(=O)$	1.1804
FSH	$(F-)(-SH)$	1.1697
		1.1653
		1.1574
CH_3Cl	$(CH_3-)(-Cl)$	1.1219
C_2HClO_2	$(=C<)(=CH-)(-Cl)(=O)_2$	1.1207

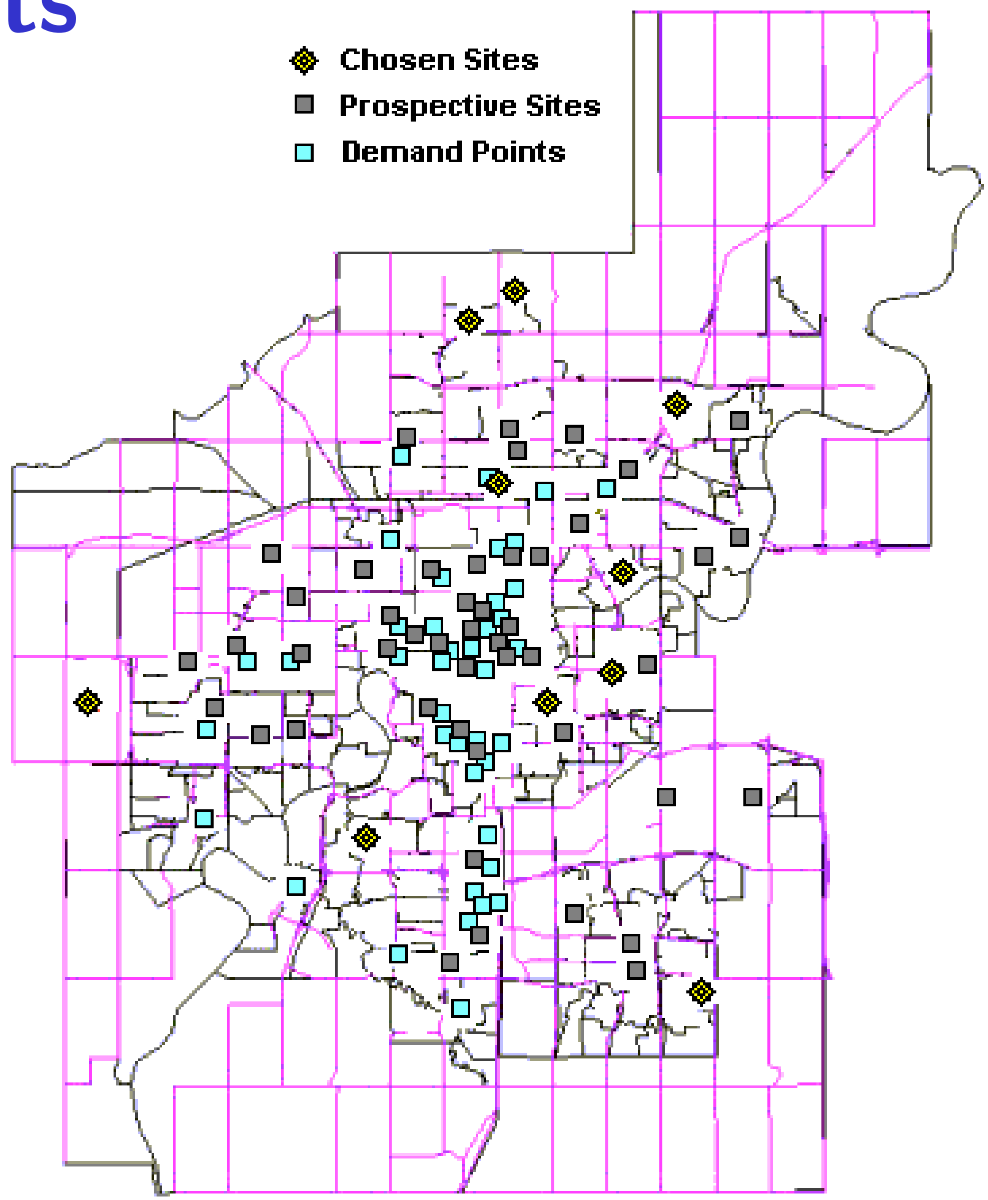
	Molecular Structure	$\frac{\Delta H_{ve}}{C_{pla}}$
CH_2FN	$(F-)(-N=)(=CH_2)$	1.1177
ClF	$(Cl-)(-F)$	1.0480
CClFO	$(O=)(=C<)(-Cl)(-F)$	1.0179
		0.9893
ClFO	$(Cl-)(-O-)(-F)$	0.9822
		0.9342
C_3H_4	$(CH_3-)(-C\equiv)(\equiv CH)$	0.9283
C_2F_2	$(\equiv C-)_2(-F)_2$	0.9229
C_3H_6O	$(-CH_3)_2(=C<)(=O)$	0.8978
C_3H_3FO	$(O=)(=CH-)_3(-F)$	0.8868
CF_2NHO	$(O=)(=C<)(-F-)_2(>NH)$	0.8763
C_2H_6	$(-CH_3)_2$	0.8632
$CHFO_3$	$(O=)(=CH-)(-O-)_2(-F)$	0.8288
$C_3H_3F_3$	$({}_{r} > CH - {}^{r})_3(-F)_3$	0.5977

All 44 feasible molecules identified

For CCl_2F_2, $\Delta H_{ve}/C_{pla} \approx 0.57$

Located Restaurants

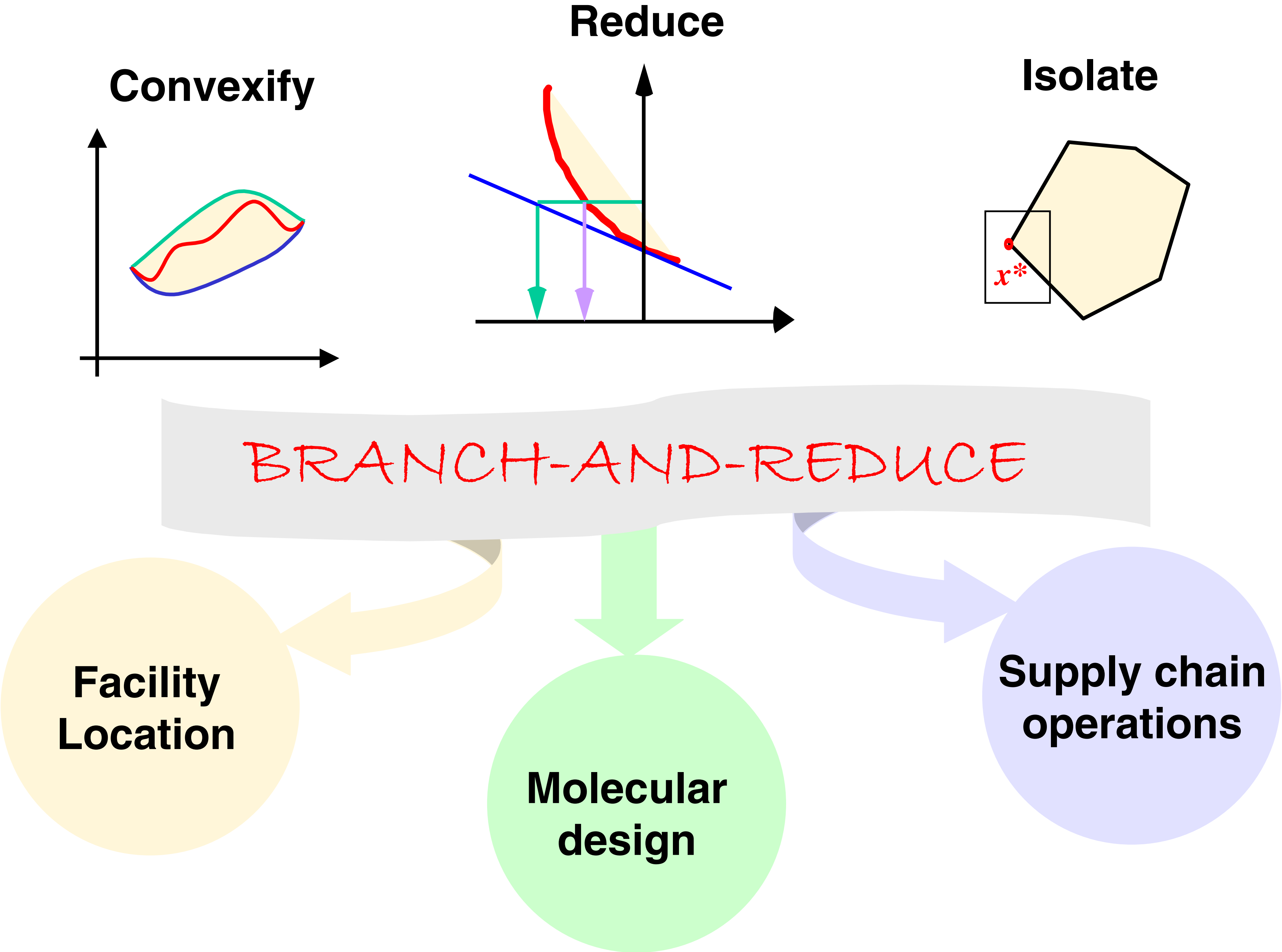
- IBM SP/2 Single Processor
- Time: 5.5 hours
- Relaxations take 95% time
- Iterations: 79
- Nodes in Memory: 9



Miscellaneous Problems

Problem	m	n	N_{tot}	N_{mem}	T_{tot}
Bilinear Problem	1	2	1	1	0.00
Design of Water Pump	3	3	1	1	0.00
Alkylation Process Design	7	10	259	27	14.8
Design of Insulated Tank	2	4	15	4	0.10
HENS	3	5	349	40	1.80
Chemical Equilibrium	3	3	1	1	0.00
Pooling Problem	7	10	24	7	0.20
Quadratic	2	2	17	6	0.10
Bilinear	1	2	21	5	0.10
Nonlinear Equality	1	2	14	2	0.10
Economies of Scale	2	3	1	1	0.00
Two-Stage Process Systems	3	4	1	1	0.00
MINLP, Process Synthesis	2	2	1	1	0.00
MINLP, Process Synthesis	9	7	9	5	0.20
MINLP, Process Synthesis	5	5	1	1	0.00
HENS	9	12	53	14	0.90
Reinforced Concrete Beam	2	1	38	13	0.10
Quadratic Constraints	4	2	28	5	0.10
QCQP	2	2	70	9	0.40
Reactor Network Design	7	5	178	22	1.40
Two-Stage Process System	6	6	1	1	0.00

Problem	m	n	N_{tot}	N_{mem}	T_{tot}
Concave QP	5	2	3	2	0.00
Biconvex Program	2	2	35	7	0.10
Linearly Constrained QP	4	2	1	1	0.00
Linear Multiplicative	8	4	1	1	0.00
Concave QP	4	2	1	1	0.00
Nonlinear Fixed Charges	6	4	3	2	0.00
QCQP	3	5	9	2	0.10
Reliability	7	11	13	5	0.1
Fixture Design	15	14	109	21	0.70
Fixture Design	133	61	1611	194	18.5
Molecular Design	18	35	1058	170	22.9
Bilinearly Constrained	6	9	16	5	0.10
Bilinearly Constrained	4	6	1	1	0.00
HENS	39	32	228	41	8.90
MINLP	2	2	145	47	1.10
Parameter Estimation	1	4	226	20	5.10
Pressure Vessel Design	3	4	13	2	0.60
Shekel Function	0	2	5	2	0.10
Truss Design	9	4	23	6	0.20
Reliability	1	4	9	4	0.10
Design of Experiments	2	5	1	1	0.00


Reduce
Convexify
Isolate
x*
BRANCH-AND-REDUCE
Facility Location
Molecular design
Supply chain operations